Conditional Probability of Hitting Barrier

A model is developed for evaluating the conditional probability of hitting an upper barrier before a lower barrier, and vice versa, for a tied down geometric Brownian motion with drift. The method produces an analytical value for this probability, assuming that the barrier levels are constant and continuously monitored.

Let *St* denote the price at time equal to *t* of an underlying security. Furthermore assume that the process {*S t* |*t* Î[0,+¥) } satisfies, under some measure *P* , the stochastic differential equation

$$dS_t = S_t\left(\mu dt + \sigma dW_t\right), \qquad t \in [0,+\infty) .$$

Also let

· *Hu* and *Hd* (where *H H u d* >) respectively denote constant upper and lower barrier levels,

· t 1 = inf { } inf *t* ³ 0|,*S* ³ *H t u* and t { } 2 = inf *t* ³ 0|*S* £ *H t d* respectively denote the first hitting times of the barrier levels *Hu* and *Hd* (here we assume that *H S H d u* < < 0), and

· *T* (where *T* > 0) denote a length of time.

We consider the conditional probability that the upper barrier level is crossed during the interval [0,*T*], and for a smaller time than for which the lower barrier level is crossed, given that *S y T* = , that is,

$$P\left(\tau_1 \le T, \tau_1 < \tau_2 \middle| S_T = y\right).$$

An analytical value for this conditional probability is provided in [Myint, 1997]. The derivation is based, in part, on an application of Theorem 4.2 in [Anderson, 1960] (see page 175), which gives an analytical value for a similar conditional probability but with respect to standard Brownian motion (see https://finpricing.com/lib/FxForwardCurve.html)

We first introduce some notation. Specifically let

$$\tau_1^{f,\gamma} = \inf\left\{t \ge 0 \middle| f(t) \ge \gamma\right\}$$

and

$$\tau_2^{f,\gamma} = \inf\left\{t \ge 0 \middle| f(t) \le \gamma\right\}$$

denote first hitting times, respectively from below and from above, of the constant barrier level g .

Next let

· {*W t* } *t* | Î[0,+¥) denote standard Brownian motion under a probability measure *P* ,
And

· g 1 and g 2 (where g 1 > 0 and g 2 < 0) respectively denote constant upper and lower barrier levels.

For standard Brownian motion, consider the conditional probability that the upper barrier level is crossed during the interval [0,*T*], and for a smaller time than for which the lower barrier level is crossed, given that *W y T* = , that is,

$$P\left(\tau_1^{W,\gamma_1} \le T, \tau_1^{W,\gamma_1} < \tau_2^{W,\gamma_2} \middle| W_T = y\right).$$

From Theorem 4.2 in [Anderson, 1960] (with d d 1 2 = = 0), this conditional probability is equal to

$$\begin{cases} \sum_{n=1}^{+\infty} \left[e^{\frac{-2}{T}\left(n^2\gamma_1(\gamma_1-y)+(n-1)^2\gamma_2(\gamma_2-y)-n(n-1)\left[\gamma_1(\gamma_2-y)+\gamma_2(\gamma_1-y)\right]\right)} - e^{\frac{-2}{T}\left[n^2\left(\gamma_1(\gamma_1-y)+\gamma_2(\gamma_2-y)\right)-n(n-1)\gamma_1(\gamma_2-y)-n(n+1)\gamma_2(\gamma_1-y)\right]} \right], & \text{if } y \le \gamma_1, \\ 1-\sum_{n=1}^{+\infty} \left[e^{\frac{-2}{T}\left((n-1)^2\gamma_1(\gamma_1-y)+n^2\gamma_2(\gamma_2-y)-n(n-1)\left[\gamma_1(\gamma_2-y)+\gamma_2(\gamma_1-y)\right]\right)} - e^{\frac{-2}{T}\left(n^2\left[\gamma_1(\gamma_2-y)+\gamma_2(\gamma_2-y)\right]-n(n+1)\gamma_1(\gamma_2-y)-n(n-1)\gamma_2(\gamma_1-y)\right)} \right], & \text{if } y \ge \gamma_1. \end{cases}$$

Also for standard Brownian motion consider the probability that the upper barrier level is crossed during the interval [0,*T*], and for a smaller time than for which the lower barrier level is crossed, and that *WT* lies in an interval *I* , that is,

$$P\left(\tau_1^{W,\gamma_1} \le T, \tau_1^{W,\gamma_1} < \tau_2^{W,\gamma_2}, W_T \in I\right).$$

From Bayes' Theorem and (1), this probability is equal to

$$\int_I g(y) P\left(\tau_1^{W,\gamma_1} \le T, \tau_1^{W,\gamma_1} < \tau_2^{W,\gamma_2} | W_T = y\right) dy$$

For the process {*S t* } *t* | Î[0,+¥) , consider the conditional probability that the lower barrier level is crossed during the interval [0,*T*], and for a smaller time than for which the upper barrier level is crossed, given that *S y T* = , that is,

$$P\left(\tau_2^{S,\gamma_2} \leq T, \tau_2^{S,\gamma_2} < \tau_1^{S,\gamma_1} \middle| S_T = I\right).$$

FP chooses to obtain this conditional probability by considering the identity

$$\begin{aligned}P\left(S_T \in I\right) = {} & P\left(\tau_1^{S,\gamma_1} \leq T, \tau_1^{S,\gamma_1} < \tau_2^{S,\gamma_2}, S_T \in I\right) \\ & + P\left(\tau_2^{S,\gamma_2} \leq T, \tau_2^{S,\gamma_2} < \tau_1^{S,\gamma_1}, S_T \in I\right) \\ & + P\left(\tau_1^{S,\gamma_1} > T, \tau_2^{S,\gamma_2} > T, S_T \in I\right),\end{aligned}$$

Given its similarity to the result for hitting an upper barrier before a lower barrier, we would like to recommend that this approach be considered for use in a future implementation of this method to price an actual deal.